Special Topics: Probability– Final Examination

Date : 24th April, 2025. Time : 10 AM - 1 PM.

Total Marks: 50   Duration: 3 Hours

All ten questions carry 5 points. If you are using results from notes, please quote/cite them appropriately. Results from a question can be used to answer later questions.

The aim of this series of questions will be to verify cutoff under product condition when mixing is considered with respect to L^p-norm for $1 < p < \infty$.

Let $X = (X_t)_{t\geq 0}$ be a discrete time irreducible Markov chain on a countable state space S with the stationary distribution π. For all $t \geq 0$, all bounded measurable $f : S \to \mathbb{R}$ and all $x \in S$, let

$$(P_t f)(x) := \mathbb{E}(f(X_t)|X_0 = x).$$

For $\mu \in \mathcal{P}$, the set of probability measures on S and $t \geq 0$, define

$$\mu_t := \mu P_t,$$

where

$$\int f(x)(\mu P_t)(dx) := \int (P_t f)(x)\mu(dx).$$

Define for $p \in [1, \infty]$,

$$\|f\|_p := \left[\int |f(x)|^p \pi(dx)\right]^{\frac{1}{p}},$$

with $\|f\|_\infty$ taken to be the essential supremum of f. And say $f \in L^p$ if $\|f\|_p < \infty$. Thus we have that

$$\|\frac{\mu}{\pi} - 1\|_p = \left[\int \left|\frac{\mu(x)}{\pi(x)} - 1\right|^p \pi(dx)\right]^{\frac{1}{p}}.$$

1 Basic properties of Norms

1. Show the following (i) $P_s \circ P_t = P_{t+s}$ for all $s, t \geq 0$; (ii) if $X_0 \sim \mu$ then $X_t \sim \mu_t$; (iii) $\pi(s) > 0, \forall s \in S$.

2. Prove that $f \in L^p$ implies that $(P_t f) \in L^p$ and $\|P_t\|_{p,p} = 1$, where $\|\cdot\|_{p,p}$ is the operator norm on L^p.

3. Show that for $q^{-1} + p^{-1} = 1$, we have

$$\|\frac{\mu}{\pi} - 1\|_p = \sup_{\substack{g\in L^q \\ \|g\|_q=1}} (\mu - \pi)(g).$$

④ Verify that

$$\|\frac{\mu}{\pi} - 1\|_1 = 2\|\mu - \pi\|_{TV} \text{ and } \|\frac{\mu}{\pi} - 1\|_2^2 = \mathrm{Var}_\pi(\frac{\mu}{\pi}).$$

⑤ Show that for any probability measure μ the function $t \mapsto \|\frac{\mu_t}{\pi} - 1\|_p$ is non-increasing.

2 Bounds on L^p-Mixing time

Define the spectral gap to be the largest constant $\lambda \geq 0$ such that for all $t \geq 0$ and $f \in L^2$

$$\|P_t f - \pi(f)\|_2 \leq e^{-\lambda t}\|f\|_2,$$

where $\pi(f) = \int f d\pi$. **We assume now onwards that the chain is ergodic i.e., $\lambda > 0$.**

Let p_0, p_1 be two numbers such that $1 \leq p_0 < p_1 \leq \infty$. Then for $0 < \theta < 1$, define p_θ by:

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

⑥ Prove that $\|f\|_{p_\theta} \leq \|f\|_{p_0}^{1-\theta}\|f\|_{p_1}^{\theta}$ for all $f \in L^{p_0} \cap L^{p_1}$.

The Riesz-Thorin interpolation theorem gives the following: For a linear operator $T : L^{p_0} \cap L^{p_1} \to L^{p_0} \cap L^{p_1}$ that boundedly maps L^{p_0}, L^{p_1} into L^{p_0}, L^{p_1} respectively, it holds that

$$\|T\|_{p_\theta} \leq \|T\|_{p_0}^{1-\theta}\|T\|_{p_1}^{\theta},$$

where $p_0, p_1, p_\theta, \theta$ are as in the previous question.

⑦ Show that for all $1 \leq p \leq \infty$,

$$\|P_t - \pi\|_{p,p} \leq C_p e^{-\lambda c_p t},$$

where $C_p := 2^{|1-2/p|}$ and $c_p = 1 - |1 - 2/p|$. *Hint: Start with $p = 1, 2, \infty$.*

⑧ For $q^{-1} + p^{-1} = 1$, show that

$$\|\frac{\mu_{t+s}}{\pi} - 1\|_p \leq C_q \|\frac{\mu_t}{\pi} - 1\|_p \, e^{-\lambda c_q s}.$$

3 Product condition implies L^p-Cutoff

Let $d(t) := \sup_{\mu \in \mathcal{P}} \|\frac{\mu_t}{\pi} - 1\|_p$. Define for $p \in [1, \infty)$ and $\eta > 0$

$$T_{mix} := T_{mix}(p, \eta) := \inf\{t > 0 : d(t) \leq \eta\}.$$

Let $1 < p < \infty$**.** Consider a sequence of processes $X^{(n)}$ on state space $S^{(n)}$ with corresponding stationary distributions $\pi^{(n)}$. Assume that the product condition is satisfied for some $\eta > 0$:

$$\lim_{n\to\infty} \lambda_n T_{mix}^{(n)} = \infty,$$

where λ_n is the spectral gap for $X^{(n)}$ and $T_{mix}^{(n)} = T_{mix}^{(n)}(p, \eta)$ for some fixed $\eta > 0$.

We will show that the cutoff phenomenon occurs under the product condition i.e., show that the following hold

(9) $\sup_{t > (1+\epsilon)T_{mix}^{(n)}} d(t) \to 0,$

(10) and $\inf_{t < (1-\epsilon)T_{mix}^{(n)}} d(t) \to \infty.$